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EQUILIBRIUM ANALYSIS: THE STABILITY THEORY
OF POINCARÉ–LIAPOUNOFF AND EXTENSIONS

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Summary: This paper will appear as an appendix to a chapter in a book by E. F. Beckenbach entitled, "Mathematics for Engineers," to be published by McGraw-Hill, 1954. This appendix sketches briefly some of the most important aspects of a part of the general theory of differential equations, the stability theory of equilibrium states.

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§1. Introduction.

The preceding chapter has treated the subject of linear and nonlinear oscillations, a topic of the utmost importance in the theory and application of differential equations. In this appendix we shall attempt to sketch briefly some of the most important aspects of another part of the general theory of differential equations, the stability theory of equilibrium states.

The fundamental problem may be posed very simply. Let us suppose that we have a physical system described at any time t by a set of state variables $(x_1(t), x_2(t), \cdots x_N(t))$. These variables are taken to vary with time in accordance with a system of differential equations,

$$(1) \qquad dx_i/dt = f_i(x_1, x_2, \cdots, x_N), \quad x_i(0) = c_i, \quad i=1,2,\cdots,N.$$

By an equilibrium state, we mean a set of values $(a_1, a_2, \cdots, a_N)$, which possesses the property that

(2) $f_i(a_1, a_2, \cdots a_N) = 0, \quad i = 1, 2, \cdots N.$

These values furnish a point solution to (1). Without the intervention of an external force, the system will remain in the state specified by $x_i = a_i$, if it starts in this state.

Suppose now that some external force, of either deterministic or stochastic origin, is applied with the result that $(a_1, a_2, \cdots a_N)$ is displaced to $(a_1', a_2', \cdots, a_N')$. Will the system return to its equilibrium position?

If it does so under all possible perturbations, of arbitrary magnitude, the equilibrium position is said to be <u>totally stable</u>. If the system returns to its equilibrium position under perturbations of sufficiently small magnitude, the equilibrium position is said to be <u>stable</u>.

It is clear that total stability is a mathematical fiction, and that in physical problems the magnitude of the disturbance will play an essential role in determining the subsequent behavior of the system.

In the succeeding sections we shall discuss the most important result in the stability theory of nonlinear differential equations, and then pass lightly over some related questions in the theory of linear differential equations, differential-difference equations, and parabolic partial differential equations.

Further results in stability theory and asymptotic behavior of differential equations will be found in the author's book [1], and previous survey [2]. A survey of results in the theory of processes involving time-lags, which lead to differential-difference

equations will be found in [3], where earlier results of E. M. Wright and the author are cited. Recent important work in the theory of nonlinear parabolic equations is due to G. Prodi [9], and [10], and earlier work is summarized in a paper by the author [4].

§2. The Stability Theory of Poincare and Liapounoff.

Let us assume, as is true in the formulation of most physical problems, that each function $f_i(x_1, x_2, \cdots, x_N)$ is a power series in the variables $x_1, x_2, \cdots, x_N$, and take, without loss of generality, each a_i to be zero.

The basic system of differential equations has then the form

$$(1) \qquad dx_i/dt = \sum_{j=1}^{N} a_{ij} x_j + g_i(x_1, x_2, \cdots, x_N),$$

where $A = (a_{ij})$ is a constant matrix and each $g_i(x)$ is a power series containing no constant or first-order terms.

Let us take as initial conditions

$$(2) \qquad x_i(0) = c_i, \quad i=1,2,\cdots,N,$$

where we assume that the c_i are chosen to be small enough so that each $g_i(x)$ is small compared to the linear terms.

It is now plausible that the stability of the equilibrium position $x_1 = x_2 = \cdots = x_N = 0$ for (1) will depend upon whether or not this state is stable for the linear approximation

$$(3) \qquad dx_i/dt = \sum_{j=1}^{N} a_{ij}x_j, \quad i=1,2,\cdots,N.$$

That this is indeed so is the substance of the classical result of Poincaré and Liapounoff.

Before presenting a formal statement of the theorem, let us consider the asymptotic behavior of the solutions of (3). As is well known, cf. [1], we may obtain all solutions of (3) in terms of simple solutions of the type $x_i = e^{\lambda t}c_i$, together with limiting forms. Substitution in (3) yields the algebraic system

$$(4) \qquad \lambda c_i = \sum_{j=1}^{N} a_{ij}c_j, \quad i=1,2,\cdots,N.$$

Elimination of the c_i, which cannot all be zero, yields the determinantal equation $|A-\lambda I| = 0$, the characteristic equation of A. The roots of this equation are called the characteristic roots of A. If multiple roots occur, we may obtain solutions of the form $x_i = e^{\lambda t}p_i(t)$, where the $p_i(t)$ are polynomials in t.

In any case, we see that the asymptotic behavior of the solutions of the linear differential equation is determined by the algebraic character of the characteristic roots. It is clear that a necessary and sufficient condition that all solutions of (3) tend to zero as $t \longrightarrow \infty$ is that all characteristic roots have negative parts.

This condition can be tested without explicit calculation of the roots by use of the Hurwitz criteria, cf. [3], once the characteristic equation, $|A-\lambda I| = 0$, has been written out in polynomial form.

Let us now state the fundamental result of Poincaré and Liapounoff:

Theorem 1.* Consider the system of differential equations appearing in (1). Let us assume that

(4)
a. all characteristic roots of A have negative real parts,
b. $|g_i(x_1, x_2, \cdots, x_N)/(|x_1|+|x_2|+\cdots+|x_N|) \longrightarrow 0$ as $|x|+|x_2|+\cdots+|x_N| \longrightarrow 0$.

Then any solution of (1) for which $|c_1| + |c_2| + \cdots + |c_N|$ is sufficiently small approaches zero as $t \longrightarrow \infty$.

This theorem affirms the correctness of using the linear approximations to test the stability of an equilibrium position. Several proofs, each of independent interest, will be found in [1].

§3. Stability Theory of Linear Equations.

The above mathematical formulation of the behavior of a system assumed stationarity, which is to say that the laws governing the behavior of the system were taken to be invariant in time. In many applications this is not the case.

Let us consider a situation where the system is governed by a system of linear equations with variable coefficients,

$$(1) \qquad dx_i/dt = \sum_{j=1}^{N} (a_{ij}+b_{ij}(t))x_j, \quad i=1,2,\cdots,N.$$

It is reasonable to suspect that if $b_{ij}(t) \longrightarrow 0$ as $t \longrightarrow \infty$, the asymptotic behavior of the solution of (1) will mimic the

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* The original result of Poincaré and Liapounoff assumed that the $g_i(x)$ were power series in the x_i possessing no constant or first-order terms. As stated above, the result is due to Perron.

behavior of the solution of the approximate equation

$$(2) \qquad dx_i/dt = \sum_{j=1}^{N} a_{ij}x_j, \quad i=1,2,\cdots,N.$$

Unfortunately for engineers, and fortunately for mathematicians, this is not universally true. Let us give a simple example. It can be shown that not all solutions of

$$(3) \qquad d^2x/dt^2 + (1 + \sin 2t/t)x = 0,$$

are bounded, despite the fact that the solutions of the approximating equation, $d^2x/dt^2 + x = 0$, are.

We have here a typical problem in the theory of the stability of the properties of an equation under perturbations of the form of the equation.

A useful result due to Hukuwara, see [1], is the following:

Theorem 2. _If_

$$(5) \qquad \begin{aligned} &\text{a.} \quad \underline{\text{all solutions of}}\ (2)\ \underline{\text{are bounded as}}\ t \longrightarrow \infty, \\ &\text{b.} \quad \int^{\infty} |b_{ij}(t)|\,dt < \infty, \quad i,j=1,2,\cdots N, \end{aligned}$$

then all solutions of (1) _are bounded._

If more is known concerning the $b_{ij}(t)$, then correspondingly more can be determined concerning the solutions. A case of particular importance is that where each $b_{ij}(t)$ possesses an asymptotic series of the form

$$(6) \qquad b_{1j}(t) \sim b_{1j0} + b_{1j1}/t + b_{1j2}/t^2 + \cdots .$$

In this case, asymptotic series for the solutions may be found. A further discussion of this very important aspect of the theory of asymptotic behavior will be found in [1], where further references may be found.

§4. Differential-difference Equations.

In many important applications involving automatic control, an appreciable time is required for the controlling mechanism to react. In problems of this type, in place of a differential equation we obtain a differential-difference equation, and occasionally a functional equation of more complicated type.

Thus, for example, in describing the motion of a damped pendulum, in place of the traditional equation,

$$(1) \qquad x''(t) + ax'(t) + bx(t) = 0,$$

we obtain the equation

$$(2) \qquad x''(t) + ax'(t-\tau) + bx(t) = 0,$$

if the damping control possesses a time lag τ.

The characteristic equation corresponding to (1), obtained by setting $x = e^{\lambda t}$, is

$$(3) \qquad \lambda^2 + a\lambda + b = 0.$$

If a and b are positive quantities, it is trivially seen that every solution of (1) goes to zero as $t \longrightarrow \infty$. The equation obtained from (2) in the corresponding fashion is

$$(4) \qquad \lambda^2 + a\,e^{-\lambda\tau} + b = 0.$$

It is no longer a trivial matter to determine the regions of the (a,b) plane which yield equations with the property that all the roots have negative real parts.

An important paper attacking this problem is by Pontrjagin [8], An English summary of this, together with some applications of the methods contained in the paper, will be found in [3]. An independent approach to a particular equation will be found in Hayes [6].

The stability theory of nonlinear differential-difference equations was inaugurated by the author and E. M. Wright, [11], who has investigated these questions in considerable detail. A bibliography will be found in [3].

Finally, we note that the theory of periodic solutions of differential-difference equations has been discussed by Minorsky [7] and by Brownell [5].

§5. The Heat Equation.

Stability problems of entirely analogous type arise in connection with the nonlinear heat equation,

$$(1) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + g(u),$$

where the boundary and initial conditions are of the form

(2) $u = 0$ on B, the boundary of a region $R(x,y,z)$,

$u = f(x,y,z)$ in R at $t = 0$.

From physical considerations it is to be expected that the conditions,

(3)
a. $\max_R |f|$ sufficiently small,

b. $g(u) = o(u)$ as $u \longrightarrow 0$,

will guarantee that the solution of (1) approaches zero as $t \longrightarrow \infty$, assuming that R is a bounded region with no particular oddities.

There are many ways of attacking this problem. A bibliography of previous work will be found in [4], where a solution is obtained for the case where R is a rectangular parallelepiped. A complete solution for general regions was obtained by P. Lax, in private communication, and in published form by G. Prodi [9], who has also investigated other interesting questions in this domain, cf. [10].

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